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Conformal invariant expansion and high-dimensional Painlevé integrable models

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Abstract. A conformal invariant asymptotic expansion approach to solving real nonlinear physical systems is proposed. In particular, the (3 + 1)-dimensional Kadomtsev–Petviashvili (3DKP) equation is solved approximately by using the conformal invariant asymptotic expansion. In some special cases, the approximate solutions of the 3DKP become exact. Many types of (3 + 1)-dimensional Painlevé integrable models are obtained at the same time.

1. Introduction

Integrable models in (1 + 1) and (2 + 1) dimensions have been studied deeply by many mathematicians and physicists and widely applied in many fields such as condensed matter physics [1], fluid mechanics [2], plasma physics [3], optics [4], communications [5], chemistry [6] and biology [7]. However, there is no known real significant $(n + 1)$-dimensional $(n \geqslant 3)$ nontrivial nonlinear integrable model except the '(2 + 2)'-dimensional self-dual Yang–Mills field equation [8]. Because the real physical space is (3 + 1) dimensional, many physicists and mathematicians have been trying to find some nontrivial (3 + 1)-dimensional integrable models [8, 9] for a long time. However, there has been little progress in this direction.

Recently, one of the present authors (Lou) proposed some possible directions to find some nontrivial integrable models in higher dimensions under some special conditions. According to the fact that all the known (2+1)-dimensional integrable models possess a common generalized Virasoro-type symmetry algebra [10], a general method to obtain some integrable models under the condition that they possess the generalized Virasoro symmetry algebra [11] is proposed. It is also known that every (1+1)- and (2+1)-dimensional integrable model possesses a Schwartz form which is conformal invariant. Based on this fact, Lou pointed out that starting from a conformal invariant form is one of the most convenient ways to obtain higher-dimensional integrable models [12]. In [13, 14], the authors have extended the Painlevé analysis approach to a new form such that many higher-dimensional Painlevé integrable models can be obtained from lower-dimensional ones.

In Lou, and Lou and Xu's papers [13, 14], some important questions need to be answered further. One of the most important questions is whether those higher-dimensional Painlevé integrable models obtained from the extended Painlevé analysis have real physical significance?

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Can they describe real nonlinear physical phenomena? Generally, most (1 + 1)- and (2 + 1)-dimensional integrable models are obtained by ignoring some not very important factors from the real physical models with the same and/or higher dimensions. It suggests that (3 + 1)-dimensional integrable models may be obtained by means of some reasonable approximations from actual (3 + 1)-dimensional models. Combining the conformal invariant theory with the extended Painlevé analysis approach may be one possible way to find higher-dimensional integrable models with real physical applications.

In this paper, we would like to use the invariant Painlevé analysis approach to solve real higher-dimensional nonlinear physical problems and find some higher-dimensional Painlevé integrable models in the same dimensions which can be considered as reasonable approximations of real physical problems.

In order to make this approach clear, we take the (3 + 1)-dimensional KP equation

$$(u_t + 6uu_x + 6u_{xxx})_x + hu_{yy} + ku_{zz} = 0 \tag{1}$$

as a simple example. Equation (1) describes the dynamics of solitons and nonlinear waves in plasmas and superfluids [15–17]. When u is z independent, equation (1) is completely integrable and then many kinds of solution can be obtained from some different approaches such as inverse scattering transformation, bilinear method, etc. Because of the nonintegrability of (1), it is difficult to give some exact solutions of (1). Some authors treat it numerically [15]. Now, we use a method of conformal invariant expansion to study the (3 + 1)-dimensional nonintegrable KP equation (1) approximately and analytically.

The paper is organized as follows. In section 2, a general approximation method (conformal invariant expansion) to solve a real physical model in any dimensions is proposed. In this approach, a nonintegrable model is solved approximately by means of Painlevé integrable models in the same dimensions. Taking the (3 + 1)-dimensional KP equation as a simple example, many new higher-dimensional conformal invariant models are derived in section 3. Section 4 is devoted to discussing the Painlevé integrability of these resulting models. The approximate solutions of the (3 + 1)-dimensional KP equation are shown in section 5. Section 6 includes a summary and discussion.

2. General theory

For a given n-dimensional Nth-order PDE,

$$F(x_0 \equiv t, x_1, x_2, \ldots, x_n, u, u_{x_i}, u_{x_i x_j,\ldots}, u_{x_{i_1} x_{i_2} \ldots x_{i_N}}) = 0 \tag{2}$$

where the function F is a polynomial function of the field u and its derivatives. Expanding the solutions of (2) near the singular manifold ϕ, we should have the form

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha} \tag{3}$$

where α is a negative integer. Substituting (3) into (2), we can obtain a recursion relation to determine the expansion coefficients u_j

$$G(j)\, u_j = f_j(x_i, \phi_{x_i}, \phi_{x_{i_1} x_{i_2}}, \ldots, u_1, u_2, \ldots, u_{j-1}) \equiv f_j \tag{4}$$

where $G(j)$ is a polynomial function j. Equation (2) possesses the Painlevé property, if $G(j) = (j+1)(j-j_1)(j-j_2)\cdots(j-j_{N-1})$, where $j_1, j_2, \ldots j_{N-1}$ are all positive integers and the resonance conditions

$$f_{j_i} = 0 \qquad (i = 1, 2, \ldots, N-1) \tag{5}$$

are satisfied identically. If equation (2) is not Painlevé integrable, equation (4) could still be taken as a recursion formula to obtain some special solutions of (2). In this paper, we would like to obtain some approximate solutions of (2) and integrable models corresponding to (2) with the help of the recursion relation (4). Because the expansion (3) is valid only for ϕ small, we ignore those terms higher than $\phi^{M+\alpha}$ and fix ϕ by $u_M = 0$, that is to say, the truncated expansion

$$u = \sum_{j=0}^{M} u_j \phi^{j+\alpha} \qquad u_M = 0 \tag{6}$$

can be considered as an approximate solution of equation (2) up to $\phi^{M+\alpha}$. Obviously, the larger the M is, the better the exactitude of the solution (6) would be. The expansion coefficients u_j in (6) can be determined from (2). Substituting equation (6) into (2), one can obtain an equation in the power form of ϕ, say,

$$\phi^{-K}\left[\sum_{j=0}^{M} F_j \phi^j + \mathrm{O}\left(\phi^{M+1}\right)\right] = 0 \tag{4$'$}$$

where the positive integer K is determined by the leading order of the PDE (2) and F_j $(j = 0, 1, 2, \ldots, M)$ are polynomial functions of u_j $(j = 0, 1, 2, \ldots, M)$ and the derivatives of u_j and ϕ. Solving $F_j = 0$, $j = 0, 1, 2, \ldots, M$ for u_j which are equivalent to those given by (4), we know that the expansion coefficients u_j are functions of the derivatives of ϕ and $J \leqslant N-1$ arbitrary functions which are some of $u_{j_1}, u_{j_2}, \ldots, u_{j_{N-1}}$. If the original PDE (2) possesses the Painlevé property, then J, the number of the arbitrary functions u_j, is equal to $N-1$. Otherwise $J < N-1$. So if $J = N-1$ and $M > j_{N-1}$,

$$u_M = 0 \tag{7}$$

shows an equation of ϕ and the arbitrary functions u_{j_1}, $u_{j_2}, \ldots, u_{j_{N-1}}$. The solution of equation (7) determines the concrete form of (6) which can be considered as an approximate solution of (2) though some possible arbitrary functions have been included because the original equation (2) is correct up to ϕ^{M+1}th order (see equation (4$'$)). Usually, N arbitrary functions are allowed to be included in the general solution of an Nth-order PDE. In our approximate solution, only $N-1$ arbitrary functions have been included because another arbitrary function ϕ has been fixed by (7). If we take $M < j_{N-1}$, there will be fewer arbitrary functions in our approximate solution.

If the original PDE (2) is not Painlevé integrable, then $J < N-1$ which means $N-1-J$ resonance conditions of (4) are not satisfied identically and then $N-1-J$ additional conditions are introduced for the J arbitrary functions and ϕ. In this case we can still obtain some approximate solutions of (2) by solving these $N-1-J$ additional conditions and (7) for J arbitrary functions and ϕ.

It is worth pointing out that after substituting (6) into (2), one can find that there may be a special selection

$$u_0 = u_1 = \cdots = u_{-\alpha-1} = 0$$

for our approximate solution that means the approximate solution is analytical with respect to the manifold $\phi = 0$. In this case we can always select $u_{-\alpha}$, $u_{-\alpha+1}, \ldots u_{-\alpha+N-1}$ as arbitrary constants, and fix other coefficients by the recursion relation and the approximate solution of (2) reads

$$u = \sum_{j=0}^{\alpha+M} u_{-\alpha+j}\phi^j \qquad u_M = 0.$$

Finally, it is worth pointing out that if we want to obtain some special solutions of (2) with more exactitude, we may do so by fixing the arbitrary functions (and constants) in two different ways.

(a) The J arbitrary functions may be fixed from (4) by $u_{M+1} = u_{M+2} \ldots = u_{M+J} = 0$ (or equivalently from (4′) by $F_{M+1} = F_{M+2} \ldots = F_{M+J} = 0$ with $u_{M+1} = u_{M+2} \ldots = u_{M+J} = 0$) such that the equation (4′) and then (2) is valid up to the ϕ^{M+J}th order and the related approximate solution is valid up to the $\phi^{M+J+\alpha}$th order.

(b) The J arbitrary functions may be fixed appropriately from (4′) such that the truncated solution (6) becomes an exact solution of the original PDE (2). More details on these points will be seen from the later concrete example.

Because there are many special interesting properties for integrable models, we hope that equation (7) is integrable at least under some special conditions. To our knowledge, all of the known Painlevé integrable models possess conformal invariant forms. Therefore, if equation (7) is conformal invariant, it may be a candidate for Painlevé integrable equation. Because of the singular manifold ϕ being arbitrary, it is possible to make (7) conformal invariant. Actually, in (1 + 1)-dimensions, Conte had changed the expansion function ϕ as $\xi = ((\phi_x/\phi) - (\phi_{xx}/2\phi_x))^{-1}$ such that the expansion coefficients are conformal invariant [18]. Similarly, we can exchange the expansion function ϕ for

$$\xi = \left(\frac{\phi_{x_n}}{\phi} - \frac{\phi_{x_n x_n}}{2\phi_{x_n}} \right)^{-1} \tag{8}$$

where x_n expresses any one of the variables $(x_0, x_1, x_2, \ldots, x_n)$, while the corresponding approximation expansion (6) is changed to

$$u = \sum_{j=0}^{M} u'_j \xi^{j+\alpha}. \tag{9}$$

From equation (8), one can easily prove the following identities:

$$\xi_{xi} = P_i - P_{i x_n}\xi + \tfrac{1}{2}(P_i S + P_{i x_n x_n})\xi^2 \qquad i = 0, 1, 2, \ldots, n \tag{10}$$

where the functions

$$P_i \equiv \frac{\phi_{x_i}}{\phi_{x_n}} \qquad \text{and} \qquad S \equiv \frac{\phi_{x_n x_n x_n}}{\phi_{x_n}} - \frac{3}{2}\left(\frac{\phi_{x_n x_n}}{\phi_{x_n}}\right)^2 \equiv \{\phi, x_n\} \tag{11}$$

are all conformal invariant. In other words, P_i and S are invariant under the Möbious transformation

$$\phi \longrightarrow \frac{a + b\phi}{c + d\phi} \qquad ad \neq bc. \tag{12}$$

It is easy to see that all the expansion coefficients u'_j in (9) are conformal invariant because they are functions of P_i, S and arbitrary function $\{u'_{j_1}, u'_{j_2}, \ldots, u'_{j_{N-1}}\}$. Therefore, the equation related to the truncated expansion coefficient u'_M

$$u'_M = 0 \tag{13}$$

is conformal invariant. Whether equation (13) is integrable under the condition that it possesses the Painlevé property should be verified further by means of Painlevé analysis.

According to Lou's idea proposed in [12], the conformal invariant equation (13) may be Painlevé integrable. Whence (13) is proved to be Painlevé integrable, then equation (2) is solved approximately by a Painlevé integrable model in the same dimensions and the approximate solution (9) is valid up to $\xi^{M+\alpha}$.

3. Conformal invariant expansion for the (3 + 1)-dimensional KP equation

In this section, we use the general theory of the last section to the real physical model, $(3+1)$-dimensional KP equation. Substituting (9) into the $(3+1)$-dimensional KP equation (1), and taking

$$\xi = \left(\frac{\phi_x}{\phi} - \frac{\phi_{xx}}{2\phi_x}\right)^{-1} \tag{14}$$

we know that u of (1) can be expanded as

$$u = \sum_{j=0}^{M} u_j \xi^{j-2} \tag{15}$$

with

$$u_0 = -2P_1^2 \qquad \text{or} \qquad u_0 = 0 \tag{16}$$

by the leading-order analysis. Starting from $u_0 = -2P_1^2$, we can obtain the recursion relation about u_j

$$(j+1)(j-4)(j-5)(j-6)u_j = f_j(S, P_i, P_{ix_i}, \ldots, u_0, \ldots, u_{j-1}) \equiv f_j \tag{17}$$

where f_j is a complicated function of the indicated variables. Because all the expansion coefficients u_j in (17) are functions of P_i, S and $\{u_{j_1}, u_{j_2}, \ldots\}$, the expansion coefficients u_j are conformal invariant. The first three expressions of u_j read

$$u_1 = 2P_1 P_{1x} + 2P_{1x} \tag{18}$$

$$\begin{aligned} u_2 = -\frac{1}{6P_1^2}\big(&4SP_1^4 + 4P_1^3 P_{1xx} + P_{1x}^2 P_1^2 + 4P_1^2 P_{1xx} \\ &+ P_1 P_0 + 2P_{1x}^2 P_1 + kP_3^2 + hP_2^2 - 3P_{1x}^2 + 4P_{1xx}P_1\big) \end{aligned} \tag{19}$$

$$\begin{aligned} u_3 = \frac{1}{24P_1^4}\big(&16hP_1P_{1y}P_2 - 20hP_{1x}P_1P_2^2 + 20kP_1^2P_3P_{3x} + 16kP_1P_{1z}P_3 + 20hP_1^2P_{2x}P_2 \\ &-16hP_1P_2P_{2x} - 16kP_1P_3P_{3x} - 20kP_{1x}P_1P_3^2 - 4kP_{1x}P_3^2 - 4hP_{1x}P_2^2 \\ &+4kP_1^2P_{3z} + 4hP_1^2P_{2y} - 8P_1^2P_{0x} + 4P_1^2P_{1xxx} + 4P_1^3P_{1xxx} - 8P_1^2P_{1xx}P_{1x} \\ &+4P_{1x}^3P_1 - 16P_1P_{1xx}P_{1x} + 12P_1^3P_{0x} + 8P_1^4P_{1x}S + 12P_1^2P_{0t} + 4P_1^4P_{1xxx} \\ &+4P_1^5S_x - 12P_1^2P_0P_{1x} - 4P_{1x}P_1P_0 + 12P_{1x}^3\big). \end{aligned} \tag{20}$$

Substituting (11), (18)–(20) into (17), one can find

$$f_4 = f_5 = 0 \qquad f_6 \neq 0 \tag{21}$$

that means the $(3+1)$-dimensional KP equation (1) is not Painlevé integrable.

To obtain some approximate solutions of (1) we can take $u_M = 0$ for an appropriate M and neglect the terms higher than ξ^{M-2} for small ξ. For $M = 2$, the ϕ equation is given by $u_2 = 0$, i.e.

$$\{\phi, x\} + \frac{\phi_t}{4\phi_x} + k\frac{\phi_z^2}{4\phi_x^2} + h\frac{\phi_y^2}{4\phi_x^2} = 0 \tag{22}$$

and the corresponding approximate solution of (1) reads

$$u = \frac{u_0}{\xi^2} + \frac{u_1}{\xi}. \tag{23}$$

For $M = 3$, the function ϕ should satisfy

$$\left(\frac{\phi_t}{\phi_x} + \{\phi; x\} + \frac{h}{2}\frac{\phi_y^2}{\phi_x^2} + \frac{k}{2}\frac{\phi_z^2}{\phi_x}\right)_x + h\left(\frac{\phi_y}{\phi_x}\right)_y + k\left(\frac{\phi_z}{\phi_x}\right)_z = 0 \tag{24}$$

while the approximate solution of (1) is

$$u = \frac{u_0}{\xi^2} + \frac{u_1}{\xi} + u_2. \tag{25}$$

If substituting (9) with $u_0 = 0$ into (1), we can find $u_1 = 0$ and u_2, u_3, u_4, u_5 can be taken as arbitrary functions. Taking

$$u_2 = a_1 \qquad u_4 = a_2 \qquad u_3 = u_5 = 0 \tag{26}$$

then the equation for $u_6 = 0$ reads

$$6a_1 + 4\frac{\phi_{xxx}}{\phi_x} - 6\frac{\phi_{xx}^2}{\phi_x^2} + \frac{\phi_t}{\phi_x} + h\frac{\phi_y^2}{\phi_x^2} + k\frac{\phi_z^2}{\phi_x^2} = 0 \tag{27}$$

where a_1 and a_2 are arbitrary constants. The corresponding approximate solution of (1) becomes

$$u = a_1 + a_2\xi^2. \tag{28}$$

In the above discussions, we have fixed ξ as given by equation (14). Actually, similar results can be obtained for different selections of ξ, say, we can take $\xi = \xi_1$ as

$$\xi_1 = \left(\frac{\phi_z}{\phi} - \frac{\phi_{zz}}{2\phi_z}\right)^{-1}. \tag{29}$$

Substituting (9) with (29) into (1) and using the same procedure, we can obtain $\alpha = -2$,

$$u_0 = -2P_1^2 = -2\frac{\phi_x^2}{\phi_z^2} \qquad \text{or} \qquad u_0 = 0. \tag{30}$$

The first three expansion coefficients corresponding to $u_0 = -2(\phi_x^2/\phi_z^2)$ read

$$u_1 = -2\left(\phi_x^2\phi_{zz} - \phi_{xx}\phi_z^2\right)\phi_z^3 \tag{31}$$

$$u_2 = -\frac{1}{6\phi_z^4\phi_x^2}\left(4\phi_x\phi_z^4\phi_{xxx} - 3\phi_z^4\phi_{xx}^2 + \phi_x\phi_t\phi_z^4 + h\phi_y^2\phi_z^4 + 3\phi_x^4\phi_{zz}^2 + k\phi_z^6 - 6\phi_x^2\phi_z\phi_{xx}\phi_{zz}\right) \tag{32}$$

$$u_3 = \frac{\phi_z}{6\phi_x^4}\left(\left(\frac{\phi_t}{\phi_x} + \{\phi, x\} + \frac{h}{2}\frac{\phi_y^2}{\phi_x^2}\right)_x + h\left(\frac{\phi_y}{\phi_x}\right)_y + \frac{k}{2}\left(\frac{\phi_z^2}{\phi_x^2}\right)_x + k\left(\frac{\phi_z}{\phi_x}\right)_z\right). \tag{33}$$

For $M = 2$ or 3, the forms of approximate solutions of (1) are the same as (23) or (25), but the corresponding function ϕ is given by

$$4\phi_x\phi_z^4\phi_{xxx} - 3\phi_z^4\phi_{xx}^2 + \phi_x\phi_t\phi_z^4 + h\phi_y^2\phi_z^4 + 3\phi_x^4\phi_{zz}^2 + k\phi_z^6 - 6\phi_x^2\phi_z^2\phi_{xx}\phi_{zz} = 0 \tag{34}$$

for $M = 2$ and

$$\left(\frac{\phi_t}{\phi_x} + \{\phi, x\} + \frac{h}{2}\frac{\phi_y^2}{\phi_x^2}\right)_x + h\left(\frac{\phi_y}{\phi_x}\right)_y + \frac{k}{2}\left(\frac{\phi_z^2}{\phi_x^2}\right)_x + k\left(\frac{\phi_z}{\phi_x}\right)_z = 0 \tag{24}$$

for $M = 3$, respectively. In the same way, if we take $u_0 = 0$ for $\xi = \xi_1$, the results read $u_1 = 0$ and u_2, u_3, u_4, u_5 being arbitrary functions. Taking $u_2 = a_1$, $u_4 = a_2$ and $u_3 = u_5 = 0$, one can obtain the equation of $u_6 = 0$

$$\begin{aligned}12\phi_x^3\phi_z^2\phi_{zzx} + 18\phi_z^2\phi_x^2\phi_{zz}\phi_{xx} - 48\phi_z\phi_x^3\phi_{zz}\phi_{zx} + h\phi_y^2\phi_z^4 + 6a_1\phi_x^2\phi_z^4 + \phi_t\phi_x\phi_z^4 - 24\phi_z^3\phi_x\phi_{zx}\phi_{xx}\\ +3\phi_z^4\phi_{xx}^2 + 36\phi_z^2\phi_x^2\phi_{zx}^2 + 4\phi_x\phi_z^4\phi_{xxx} - 12\phi_x^2\phi_z^3\phi_{zxx} + k\phi_z^6 + 9\phi_x^4\phi_{zz}^2 = 0\end{aligned} \tag{35}$$

while the approximate solution of (1) is

$$u = a_1 + a_2\xi^2 \tag{36}$$

up to ξ^4, where a_1 and a_2 are arbitrary constants.

When ϕ is z (or y) independent, equation (24) is just the $(2+1)$-dimensional KP equation in its Schwarz form. If ϕ is both y and z independent then both equation (22) and (24) reduce back to the $(1+1)$-dimensional Schwarz KdV equation.

4. Painlevé integrability of resulting equations

It is clear that (22), (24), (27), (34) and (35) are all conformal invariant. In [12], Lou had pointed out that many kinds of quite general conformal invariant forms are Painlevé integrable. Now we hope to know whether the real approximate equations, (22), (24), (27), (34) and (35) are Painlevé integrable or not. Actually, equation (22) is just one of special case in [12]. So (22) is Painlevé integrable. In order to prove the Painlevé integrability of (24), (27), (34) and (35), we slightly enlarge them to form some more general systems. From equation (24), a slightly enlarged system has the form

$$C_x\left(C^4 + hV^2 + LC + kW^2 + 4CC_{xx} - 3C_x^2\right) - C^2\left(C_{xxx} + L_x + aV_y + kW_z\right) = 0 \tag{37}$$

$$C_t = L_x \tag{38}$$

$$V_t = L_y \tag{39}$$

$$W_t = L_z. \tag{40}$$

Equation (37) can be obtained directly from (24) by using the transformation,

$$\phi = \exp(f) \qquad f_x = C \qquad f_y = V \qquad f_z = w \qquad f_t = L \tag{41}$$

as in [12] and equations (38)–(40) are three consistent conditions of the transformation (41). We call (38)–(40) as an enlarged system of (24) because, for the transformation (41), there exist three other consistent conditions, $C_y = V_x$, $C_z = W_x$ and $V_z = W_y$, though the relations $C_{yt} = V_{xt}$, $C_{zt} = W_{xt}$ and $V_{zt} = W_{yt}$ are guaranteed by (38)–(40). In other words, only a subset of the solutions of (37)–(40) are related to the solutions of (24).

In the same way, the corresponding enlarged systems of (27), (34) and (35) have the forms

$$6a_1C^2 + 4CC_{xx} - 2C^4 - 6C_x^2 + LC + hV^2 + kW^2 = 0 \tag{42}$$

$$C_t = L_x \tag{43}$$

$$V_t = L_y \tag{44}$$

$$W_t = L_z \tag{45}$$

$$\begin{aligned} &CLW^4 + hV^2W^4 - 6C^2W^2C_xW_z - 2C^4W^4 + 4CW^4C_{xx} \\ &\qquad + kW^6 + 3C^4W_z^2 - 3W^4C_x^2 = 0 \end{aligned} \tag{46}$$

$$C_t = L_x \tag{47}$$

$$V_t = L_y \tag{48}$$

$$W_t = L_z \tag{49}$$

and

$$\begin{aligned} &-hV^2W^4 - LCW^4 - 6a_1C^2W^4 - kW^6 + 2C^4W^4 - 12C^3W^2C_{zz} - 18W^2C^2W_zC_z \\ &\qquad -3W^4C_x^2 + 24W^3CC_zC_x + 48WC^3W_zC_z - 9C^4W_z^2 + 12C^2W^3C_{zx} \\ &\qquad -36W^2C^2C_z^2 - 4CW^4C_{xx} = 0 \end{aligned} \tag{50}$$

$$C_t = L_x \tag{51}$$

$$V_t = L_y \tag{52}$$

$$W_t = L_z \tag{53}$$

respectively. Now using the standard Painlevé analysis, we can prove that all the enlarged systems related to (24), (27), (34) and (35) possess the Painlevé property.

After finishing the leading order analysis of (37)–(40), we know that C, L, W and V, should be expanded as

$$\begin{aligned} C &= \sum_{j=0}^{\infty} C_j \phi^{j-1} \qquad L = \sum_{j=0}^{\infty} L_j \phi^{j-1} \\ W &= \sum_{j=0}^{\infty} W_j \phi^{j-1} \qquad V = \sum_{j=0}^{\infty} V_j \phi^{j-1} \end{aligned} \tag{54}$$

with

$$C_0 = \pm\phi_x \qquad V_0 = \pm\phi_y \qquad W_0 = \pm\phi_z \qquad L_0 = \pm\phi_t. \tag{55}$$

Substituting (54) into (37)–(40), we know that the resonances occur at

$$j = -1, 1, 1, 1, 1, 2 \tag{56}$$

and the resonance conditions at $j = 1, 2$ read

$$C_0^2\left(-2\phi_x\phi_{xx}C_0 + C_{0x}\left(\phi_x^2 + C_0^2\right) + 4C_1\phi_x^3 - 4C_0C_1\phi_x\right) = 0 \tag{57}$$

$$C_{0t} = L_{0x} \tag{58}$$

$$V_{0t} = L_{0y} \tag{59}$$

$$W_{0t} = L_{0z} \tag{60}$$

$$\begin{aligned} &C_0^3\phi_{xxx} - C_0^2C_{0x}\phi_{xx} - C_0^2\phi_xC_{0xx} - 8C_1C_0^2\phi_x\phi_{xx} + C_0^3\phi_t + kC_0^2W_0\phi_z + hC_0^2V_0\phi_y \\ &\quad + C_{1x}\left(-C_0^2\phi_x^2 + C_0^4\right) + C_0C_{0x}^2\phi_x + C_{0x}\left(4C_1C_0\phi_x^2 + 4C_0^3C_1\right) \\ &\quad + \phi_x\left(\left(3C_0^2C_2 + 6C_1^2C_0\right)\phi_x^2 - 3C_0^4C_2 - 6C_1^2C_0^2 - C_0^2L_0\right. \\ &\quad \left. + \left(-kW_0^2 - V_0^2h\right)C_0\right) = 0. \end{aligned} \tag{61}$$

Obviously, (57)–(61) are satisfied identically because of (55). So the equation system (37)–(40) possess the Painlevé property, i.e. the system (37)–(40) is Painlevé integrable and then its narrowed system (24) is Painlevé integrable naturally.

Substituting (54) into (42)–(45), we can see that the resonances occur at

$$j = -1, 1, 1, 1, 1 \tag{62}$$

and the resonance conditions at $j = 1$ read

$$-4C_0\left(-C_{0x}\phi_x + 2C_0^2C_1 + C_0\phi_{xx} - 2C_1\phi_x^2\right) = 0 \tag{63}$$

$$V_{0t} - L_{0y} = 0 \tag{64}$$

$$W_{0t} - L_{0z} = 0 \tag{65}$$

$$C_{0t} - L_{0x} = 0. \tag{66}$$

It is easy to see that all the resonance conditions (63)–(66) are also satisfied. So the equation system (42)–(45) possess the Painlevé property and equation (27) is a $(3+1)$-dimensional Painlevé integrable model.

In the same way, substituting (54) into (46)–(49), one can see that all five of the needed resonances located at

$$j = -1, 1, 1, 1, 1 \tag{67}$$

have the forms

$$-2C_0W_0\big(W_0^3\phi_xC_{0x}+2W_0^3C_0\phi_{xx}-10W_0^2C_0W_1\phi_x^2-4W_0^3\phi_x^2C_1+3C_0^3\phi_zW_{0z}-6W_0C_0^2C_1\phi_z^2$$
$$-3W_0C_0^2\phi_xW_{0z}-3W_0^2C_0\phi_zC_{0x}+6W_0C_0^2\phi_z\phi_xW_1+6W_0^2C_0\phi_z\phi_xC_1$$
$$+4W_0^2C_0^3W_1+4W_0^3C_0^2C_1\big)=0 \tag{68}$$

$$C_{0t}-L_{0x}=0 \tag{69}$$

$$V_{0t}-L_{0y}=0 \tag{70}$$

$$W_{0t}-L_{0z}=0 \tag{71}$$

and they are all satisfied naturally because of (55). That is to say equation (34) is also a Painlevé integrable model. Substituting (54) into (50)–(53), we see that all five of the required resonances are also located at

$$j=-1,1,1,1,1 \tag{72}$$

and the resonance conditions at $j=1$ read

$$-15C_0^3W_{0z}\phi_z+6C_0^3W_0\phi_{zz}+24C_0^2W_0\phi_zC_{0z}-6C_0^2W_0^2\phi_{zx}+9C_0^2W_0\phi_xW_{0z}+2C_0W_0^3\phi_{xx}$$
$$-18C_0W_0^2\phi_xC_{0z}-9C_0W_0^2\phi_zC_{0x}+7W_0^3\phi_xC_{0x}$$
$$+W_1C_0\big(4C_0^2W_0^2-36C_0^2\phi_z^2+54C_0W_0\phi_x\phi_z-22W_0^2\phi_x^2\big)$$
$$+\big(4C_0^2W_0^2-18C_0^2\phi_z^2+18C_0W_0\phi_x\phi_z-4W_0^2\phi_x^2\big)C_1W_0=0 \tag{73}$$

$$C_{0t}-L_{0x}=0 \tag{74}$$

$$V_{0t}-L_{0y}=0 \tag{75}$$

$$W_{0t}-L_{0z}=0. \tag{76}$$

Having considered (55), equations (73)–(76) are all satisfied. Obviously, the equation system (50)–(53) possesses the Painlevé property. So equation (35) is also integrable under the meaning that it possesses a variant form with the Painlevé property.

Now we have obtained five new (3 + 1)-dimensional models possessing Painlevé integrability from the (3+1)-dimensional nonintegrable KP equation (1) through the conformal invariant expansion approach. The solutions of these models can be used to express the solutions of original (3 + 1)-dimensional KP equations approximately in some special ways.

5. Solitary wave solutions

To find some concrete approximation solutions of equation (1), we should solve the Painlevé integrable models (22), (24), (27), (34) and (35). Because these models are conformal invariant, it is easy to verify that these conformal invariant equations all possess kink-type soliton solutions. For model (24), the corresponding kink-type plane soliton solution has the form

$$\phi=\frac{A+B\exp 2(l_1x+l_2y+l_3z+wt)}{C+D\exp 2(l_1x+l_2y+l_3z+wt)}. \tag{77}$$

When

$$w=\frac{8l_1^4-kl_3^2-hl_2^2}{l_1} \tag{78}$$

equation (77) is also a solution of (22) and (34). The corresponding ξ is given by

$$\xi=-\frac{1}{l_1}\frac{A+B\exp 2(l_1x+l_2y+l_3z+wt)}{A-B\exp 2(l_1x+l_2y+l_3z+wt)}. \tag{79}$$

While the related approximation solution of (1) reads

$$u = -2l_1^2\frac{(A + B\exp 2(l_1x + l_2y + l_3z + wt))^2}{(A + B\exp 2(l_1x + l_2y + l_3z + wt))^2} - \frac{A_1}{6l_2^2} \tag{80}$$

where

$$A_1 = -8l_1^4 + hl_2^2 + wl_1 + kl_3^2. \tag{81}$$

The solution (80) can be written in the standard form

$$u = -\left(2l_1^2 + \frac{A_1}{6l_2^2}\right) + 2l_1^2\left(\operatorname{sech}\left(l_1x + l_2y + l_3z + wt + \ln\left(\frac{B}{A}\right)^{1/2}\right)\right)^2. \tag{82}$$

If the arbitrary constants l_2 and w are taken as

$$l_2 = \frac{(-(16l_1^2 + h)(-12l_1^4 + l_3^2k))^{1/2}}{16l_1^2 + h} \qquad w = \frac{4l_1^4 + l_2^2h + l_3^2k}{2l_1} \tag{83}$$

and u given by (82) becomes an exact solution of (1). We can also verify that

$$\phi = \frac{A - B\exp 2\big(l_1x + l_2y + l_3z + \big(-6l_1^2a_1 + 8l_1^4 - hl_2^2 - kl_3^2\big)t/l_1\big)}{A + B\exp 2\big(l_1x + l_2y + l_3z + \big(-6l_1^2a_1 + 8l_1^4 - hl_2^2 - kl_3^2\big)t/l_1\big)} \tag{84}$$

is a kink-type plane solitary wave solution of both (27) and (35). The corresponding ξ is given by

$$\xi = -\frac{2}{l_i}\frac{A - B\exp 2\big(l_1x + l_2y + l_3z + \big(-6l_1^2a_1 + 8l_1^4 - hl_2^2 - kl_3^2\big)t/l_1\big)}{A + B\exp 2\big(l_1x + l_2y + l_3z + \big(-6l_1^2a_1 + 8l_1^4 - hl_2^2 - kl_3^2\big)t/l_1\big)} \tag{85}$$

$i = 1$, for (27) and $i = 3$ for (35). In this case, the approximation solution of (1) reads

$$\begin{aligned} u &= a_1 + a_2\xi^2 \\ &= a_1 + \frac{a_2}{l_i^2}\left(1 - \operatorname{sech}^2\left(l_1x + l_2y + l_3z + \big(-6l_1^2a_1 + 8l_1^4 - hl_2^2 - kl_3^2\big)\frac{t}{l_1} + \ln\left(\frac{B}{A}\right)^{1/2}\right)\right). \end{aligned} \tag{86}$$

If the arbitrary constants a_1 and a_2 are taken as

$$a_1 = 2l_1^2 \qquad a_2 = -2l_1^2l_i^2 \tag{87}$$

u given by (86) is an exact solution of equation (1). Generally, the approximate procedure is valid for ξ being small. In the plane soliton case, ξ given by (79) is small which means u given by (80) is valid near the soliton centre. While the ξ given by (85) is small and then means that u given by (86) is only valid far away from the soliton centre.

6. Summary and discussion

In this paper we have proposed a simple method to solve higher-dimensional nonlinear problems by means of the Painlevé integrable models in the same dimensions. Taking the (3 + 1)-dimensional KP equation as a concrete example, we have obtained many (3 + 1)-dimensional Painlevé integrable models. The first one is just a special case proposed by Lou in [12]. Because the resulting equations possess conformal invariance, the plane solitary wave solutions of the model can be obtained easily. Generally, the solitary waves are approximate for the original model. If the arbitrary constants are selected suitably, the approximate solutions become exact.

In [12–14], Lou has proposed many types of conformal invariant equations and those equations are Painlevé integrable. There are two important problems left in his paper.

(a) Can one find the physical applications of those models or can one find some types of models with conformal invariance and the Painlevé property from real physical models? (b) Are those Painlevé integrable models integrable under other meanings? In this paper, we have offered a positive answer for the first question. Using the conformal invariant expansion to any real physical model in any dimensions, we can obtain some approximate solutions by means of the Painlevé models. The second question is still open. It is worth further study to see whether the models obtained here are integrable under other traditional methods.

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